A NOVEL PROCEDURE FOR SOLVING BEAM AND TRUSS PROBLEMS

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Abstract: In the present paper we present a technique for solving ordinary and partial differential equations (ODE & PDE) linear and nonlinear by an innovative method. The innovative method consists of decomposing a given differential equation into linear, nonlinear and remainder terms. The method has been applied quite extensively by mathematicians for various cases. However, engineering applications are not that many. While applying the method to a static beam and static problem we observed that the solution with innovative one very close to numerical and analytical solutions. An innovative method has been applied for linear equation beam problems to improve the simplicity, accuracy and convergence of above mentioned problems. The beam problems can easily solve with help of innovative method, which is decomposing technique and semi-analytical method. The decomposition innovative method results are found to converge very quickly and are more close to exact solution.

1. INTRODUCTION TO DECOMPOSITION INNOVATIVE METHOD

A novel technique applied for solving continuous support bridge problems. In recent years the development of the high-speed digital computer and increased interest in continuous and linear phenomena have led to an intensive study of the numerical solution of ordinary and partial differential equations. The innovative decomposition method is a non-numerical method for solving linear and nonlinear differential equations, both ordinary and partial. The general direction of the paper is towards obtaining solution for ordinary and partial differential equations (PDEs). In the 1985s, Adomian [1, 2] proposed a new and ingenious method to obtain exact solution of linear and nonlinear equations of various kinds like algebraic, differential for both ordinary and partial, integral, etc. problems. The technique uses a decomposition of the nonlinear operator as a series of Adomian functions. Each term of this series is a generalized polynomial called the Adomian polynomial. Some techniques which assume essentially that the linear and nonlinear system is almost linear after equivalent linearization will not be able to retain the originality of the problem. Present technique consists of splitting the given equation into linear, remainder and nonlinear parts, inverting the highest order differential operator contained in the linear operator on both sides, identifying the initial or boundary conditions and the terms involving the independent variable alone as initial approximation, decomposing the unknown function into a series whose components are to be determined, decomposing the linear and nonlinear function in terms of special polynomials, and finding the successive terms of the series solution. The present innovative method provides the solution in a rapidly convergent series with easily computable components. The main advantage of the method is that it can be used directly to solve, all types of differential equations with homogeneous and inhomogeneous boundary conditions. Another advantage of the method is that it reduces the computational work in a tangible manner, while maintaining higher accuracy of the numerical solution. The conventional have systematic procedure and follows some assumed rules, but using the innovative method, one can solve the problem straight way.

Recently, Morawetz[13] solved a first order nonlinear wave equation (NLWE) problem differently and applied for wave equations only. The solution obtained by this method is derived in the form of a power series with easily computable components. The decomposition method requires that the nonlinear operator F be separated into three terms as follows: F = L+R+N where N is a nonlinear operator and L + R together form the linear term. Here L is chosen to be easily invertible and R is the remainder of the linear term. To convey the idea and for the sake of completeness of the innovative decomposition method, a three dimensional time variant (x, y, z, t) equation is considered as shown below,

\[ L_xu + L_yu + L_zu + N(u) + R(u) = g(x, y, z, t) \] (1)

Where \( L_x, L_y, L_z, L_t \) are \( n^{th} \) order derivative operators. Solution for u is obtained after operation w.r.t. x,

\[ L_xu = g(x, y, z, t) - (L_yu + L_zu + L_tu + N(u) + R(u)) \] (2)

Now pre-multiplication by \( L_x^{-1} \) on both sides for equation (2)

\[ L_x^{-1}L_xu = L_x^{-1}g(x, y, z, t) - L_x^{-1}(L_yu + L_zu + L_tu + N(u) + R(u)) \] (3)

This equation \( L_x^{-1}L_x \) is multiplication of integration and differentiation of \( n^{th} \) order differential equation. Therefore it yields n constants, \( a_1, a_2, a_3 \) and \( a_n \).
Therefore the \( n \)th order equation is
\[
\Rightarrow u = a_1 + x a_2 + \cdots + x^{n-1} a_n + \frac{d}{dx} x^{n-1} a_n g(x, y, z, t) - \frac{d}{dx} x^{n-1} (L_y u + L_z u + L_t u + N(u) + R(u))
\]
(5)
Where \( a_1, a_2 \) and \( a_n \) are \( n \) constants of integration. For IVPs and BVPs, the constraints need to be found from initial and boundary conditions respectively. The boundary conditions are \( u|_{x=0}, \frac{du}{dx}|_{x=0} \), and and initial conditions are \( u|_{t=0}, \frac{du}{dt}|_{t=0} \). Here, the nonlinear part \( N(u) \) is assumed to be a contracting (nonlinear) exact operator and decomposed as an infinite sum of functions,
\[
N_n(u) = \sum_{i=0}^{n} A_i(u_0, u_1, u_2, \ldots, u_{n-1})
\]
(6)
Where are the Adomian polynomials valid only for the specific \( N(u) \). Adomian polynomials \( A_i \) depend on \( u_i \) for \( i = 0, \ldots, n-1 \) and form a rapidly convergent series. Now, let the solution \( u \) of the above equation obtained as a series of functions \( u_i, i = 0 \) to \( n \), i.e.,
\[
u = u_0 + u_1 + u_2 + u_3 + \cdots + u_n.
\]
(7)
With the preceding assumptions on \( u \) and \( N \), the Adomian series equations are solutions of basic equation. Next it is required to find Adomian polynomials of above equation, which are needed to derive a series of solutions of equation by Adomian iterative procedure. Here it may be noted that \( u_n \) is absent in the series because \( N_n \) depends up on Adomian polynomial \( A_{n-1} \), i.e. \( u_0, u_1, u_2, \ldots, u_{n-1} \).

2. AN INNOVATIVE DECOMPOSITION APPLICATIONS

In some of the problems, it may be observed that boundary conditions are defined only at one of the boundaries. It should not be construed that they are the only boundary conditions, in which may not be equilibrium. For such cases there will be obviously other boundary conditions also to satisfy equilibrium. It may also be mentioned here that specifying the boundary conditions on one boundary only is a special advantage of an innovative decomposition method. The innovative decomposition method is applied to several one and two-dimensional Laplace’s and Poisson’s problems and the accuracy and convergence characteristics of the method are investigated. In the first few examples are considering an one-dimensional problem. The first example with forcing term \( (x) \), a known exact solution by using innovative decomposition method, conventional and FEM methods.

2.1 One Dimensional Poisson’s Equation with Mixed Boundary Condition’s

The first example is a Poisson’s equation with a forcing term i.e. a function of \( x \). The governing Poisson’s equation is,
\[
\frac{d^2 u}{dx^2} + x = 0, \quad 0 \leq x \leq 1
\]
(1)
The Dirichlet and Neumann boundary conditions
\[
u(0) = 0 \quad \frac{du(1)}{dx} = 0.
\]
2.1.1 An Innovative Solution

Using the operator notation, equation (8) can be written as \( L_x u + x = 0 \), \( L_x u = -x \), where \( L_x = \frac{d^2}{dx^2} \).

Pre multiplying both sides of the equation (8) by \( L_x^{-1} \),
\[
L_x^{-1} L_x u = L_x^{-1} (-x)
\]
\[
u = a_1 + x a_2 + L_x^{-1} (-x) = a_1 + x a_2 + L_x^{-1} (-x),
\]
Using Dirichlet and Neumann boundary conditions, one can determine the unknown Constants \( a_1, a_2 \), Putting \( atx=0 \) in the above equation \( a_1=0 \), and other BC \( atx=1 \),
\[
a_2 = \frac{1}{2^2} u_{00} = a_1 = 0,
\]
Since there are no remainder nor nonlinear terms in the equation(), \( u_1 \) in next iteration does not exist, i.e.\( u_1=0 \).

\[
\begin{align*}
\vdots \\
\sum_{i=0}^{n} u_i = u_0 + u_1 + u_2 + +u_n \approx u_0.
\end{align*}
\]

The innovative decomposition method solution for

\[
u(x) = a_1 + x a_2 + x^3 - \frac{x^3}{3!} \approx x - \frac{x^3}{3!}.
\]

Similarly, an analogous problem in solid mechanics is governed by the differential equation in terms of the displacement \( u \) is

\[
AE \frac{d^2 u}{dx^2} + q(x) = 0,
\]

where \( u \) is the axial displacement of a bar subjected to axial deformation only.

\[
\text{Figure 1. Axially deformed Bar.}
\]

Since \( q(x) \approx a x \), an inhomogeneous ODE for the displacements in the bar \( AE \frac{d^2 u}{dx^2} + a x = 0 \). The mixed boundary conditions are, \( u(0) = 0 \), \( \frac{du}{dx} \bigg|_{x=L} = \frac{R}{AE} \), where \( R \) is reaction at end support. \( AE \) is axial rigidity. In the innovative decomposition method procedure, above equation(9) can be written as \( L_x u + q(x) = 0, L_x u + a x = 0 \Rightarrow L_x u = -a x \), where \( L_x = \frac{d^2}{dx^2} \). Using the Dirichlet and Neumann boundary conditions, Pre multiplying both sides of the equation (9) by \( L_x^{-1} \).

\[
\Rightarrow L_x^{-1} L_x u = L_x^{-1} \left( \frac{a x}{AE} \right) \\
\Rightarrow u = a_1 + x a_2 + L_x^{-1} \left( \frac{a x}{AE} \right)
\]

Using Dirichlet and Neumann the boundary conditions are, one can determine the unknown constants \( a_1, a_2 \), Putting \( atx=0 \) in aboveeqution \( a_1=0 \), and other BCatx=L, \( a_2 = \frac{R}{AE} + \frac{ai^2}{2AE}, \therefore u_0 = a_1 \approx 0, \)

\[
u_0 = x a_2 + L_x^{-1} \left( \frac{a x}{AE} \right),
\]

\[
u_1 = L_x^{-1} (Ru_0 + Nu_0),
\]

Since there are no remainder nor nonlinear terms in the above equation \( u_1 \) in the next iteration does not exist, i.e. \( u_1=0 \).

\[
\vdots \\
\sum_{i=0}^{n} u_i = x \left( \frac{R}{AE} + \frac{ai^2}{2AE} \right) - \frac{ax^3}{6AE}.
\]
An innovative decomposition solution for $u(x) = x\left(\frac{R}{AE} + \frac{al^2}{2AE}\right) - \frac{ax^3}{6AE}$. The displacement field in the bar is represented as $u(x) = \frac{-a x^3 + (6R + 3al^2)x}{6AE} + \frac{al^2}{2AE}$. Once the displacement known, the slope gives the strain i.e. $\frac{du}{dx}$. The strain in the bar is $\varepsilon(x) = \frac{-a x^3 + (2R + al^2)}{2AE}$. Once the strain known, the strain multiply by Young’s modulus gives the stress i.e. $E \frac{du}{dx}$. The stress in the bar is $\sigma(x) = \frac{-a x^3 + (2R + al^2)}{2A}$.

2.1.2 Exact Solution:
The exact solution[14] is $u(x) = x\left(\frac{R}{AE} + \frac{al^2}{2AE}\right) - \frac{ax^3}{6AE}$.

2.1.3 FEM Solution:
The FEM (Galerkin) solution is shown in figure (2).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Variation of Displacement, Slope, Error and Normal Stress $\sigma_y$ solution (Present technique, Conventional method, FEM & exact)}
\end{figure}
### 2.1.4 Discussion of Results:

The above problem has an important engineering application. It is a problem where in cantilever beam axial force acting at free end. The $x$-axis is chosen to vary from $x = 0$ to $x = L = 8$. Results of the present approach or Present technique are compared with those of conventional method, FEM and exact. The results of present technique, conventional method, FEM and exact displacement, slope, error and normal stress are shown in figure (2). The innovative decomposition method displacement versus $x$ plot in figure (2a) shows that cubic parabolic variation. The corresponding slope versus $x$ plot shows that quadratic parabolic variation in figure (2b). It may be observed that the results of innovative decomposition method coincide perfectly with those of exact, which FEM results both uniform and nonuniform do not match with the exact. The error however is quite small equal to $10^{-4}$. The normal stress $\sigma$ versus $x$ plot as shown in figure (2d) indicates that parabolic variation along the length of the beam.

### 2.2 A Cantilever Beam with UDL & End Moment

Considering the $4^{th}$ order beam equation

$$\frac{d^2}{dx^2} \left( b(x) \frac{d^2 u}{dx^2} \right) - f(x) = 0 \text{ for } 0 \leq x \leq L, \quad (10)$$

subjected to boundary conditions

$$u(0) = 0, \quad \frac{du}{dx} \bigg|_{x=0} = 0, \quad b(x) \frac{d^2 u}{dx^2} \bigg|_{x=L} = M, \quad \frac{db(x)}{dx} \frac{d^2 u}{dx^2} \bigg|_{x=L} = 0, \quad (11)$$

where $u$ denotes transverse displacement, $L$ is length of the beam, $b(x)$ is flexural rigidity of the beam, $M$ is end moment and $f(x)$ is transverse distributed load. Let $b(x), f(x)$ be constants $b$ and $f$ respectively, for simplicity.

#### 2.2.1 Innovative decomposition Solution:

The above equation (10) can be written as

$$b L_x^2 u = f(x),$$

where the linear operator is $L_x = \frac{d^4}{dx^4}$. Pre-multiplying both sides of the equation (10) by $L_x^{-1}$,

$$L_x^{-1} L_x u = L_x^{-1} \left( \frac{f(x)}{b} \right)$$

$$u = a_1 + x a_2 + x^2 a_3 + x^3 a_4 + L_x^{-1} \left( \frac{f(x)}{b} \right)$$

In the usual notation, $u_0 = a_1 + x a_2 + x^2 a_3 + x^3 a_4 + L_x^{-1} \left( \frac{f(x)}{b} \right)$.

Using Dirichlet and Neumann boundary conditions, one can find the unknown constants $a_1, a_2, a_3$ & $a_4$. From the first two homogeneous BCs $a_1 = a_2 = 0$.

$$u_0 = a_3 x^2 + a_4 x^3 + \frac{1}{24} \frac{f x^4}{b},$$

from next two boundary conditions, $a_3 = \frac{2 M - L^2}{4 b}, a_4 = \frac{l f}{6 b}$.

$$u_0 = \frac{1}{4} \frac{(f L^2 - 2 M) x^2}{b} + \frac{1}{6} \frac{f L x^3}{b} - \frac{1}{24} \frac{f x^4}{b}.$$
Since there are no remainder nor nonlinear terms in the above equation, \( u_1 \) in the next iteration does not exist, i.e.  
\[
 u = \sum_{i=0}^{n} u_i = u_0 + u_1 + \ldots + u_n = u_0, 
\]
\[
 u \approx \frac{1}{4} \left( \frac{f l^2 - 2 M}{b} \right) x^2 + \frac{1}{6} \frac{f l x^3}{b} - \frac{1}{24} \frac{f x^4}{b}. 
\]

The displacement at the free end i.e. \( x = L \), \( u|_{x=L} = \frac{f l^4}{8 b} + \frac{l^2 (2 M - L^2 f)}{4 b} \).

2.2.2 Exact Solution:

The exact solution[15] is 
\[
\frac{x^2 (2 M - f l^2)}{4 b} - \frac{x^4}{24 b} + \frac{x^3}{6 b}. 
\]

The displacement at the free end i.e. \( x = L \), \( u|_{x=L} = -\frac{f l^4}{8 b} + \frac{M l^2}{2 b} \).

2.2.3 FEM Solution: The FEM (Galerkin) solution is

\[
\frac{1781919733461205}{1701411834604692317316873037155884105728} x^5 + \frac{8276175723656881}{78706180479939} x^3 + \frac{5316911983139663494161228241121378304}{13258459730297947} x^2 + \frac{230584309213693952}{3062065968172525} x^3 + \frac{590295810358705651712}{40564819207303340847894502572032} x^2 + \frac{10141204801825835211973625643008}{10141204801825835211973625643008} x
\]

Here element size is 0.02.

Figure 4: Variation of Displacement, Slope error with x (Present technique, Exact and FEM).

2.2.4 Discussion of Results:

It is an engineering problem of beam with uniformly distributed load and moment at free end. The length of beam is 50m. Figure (4) shows the variation of displacement and slope along x direction, the x-axis varies from 0 to 50 units. All the above results very closely agree with one another. The error is much less, of the order of 1. Figure (4a) shows quatratic nature and f figure (4b) shows cubic variation. One can observe that, through FEM solution, descretization makes more tedious for getting convergence of element. It requires lots of experience by computationally, until unless one can get finer mesh, FEM solution convergence is very slow.
3. CONCLUSIONS
A number of problems which are generally encountered in engineering are solved using innovative decomposition method. Most of the problems have exact solutions as well as numerical solutions. Therefore the results from innovative decomposition method are compared with the exact and numerical solutions. Whenever the exact or numerical solutions are not available, the innovative decomposition method solution is compared with MATLAB solution.

In all the problems it may be observed that innovative decomposition method results perfectly agree with the exact solutions. Different types of problems have been solved in order to confirm the robustness of the method over a wide variety of second order axial bar and truss and also fourth order differential equation for beam linear problems seen effectively. In the present paper, the problems considered have important engineering applications. Therefore the results from innovative decomposition approach are compared with the exact solution. The innovative decomposition approach procedure is simple to apply for axial bar, truss and beam problems. Like analytical solution, one can get convergent solution using present novel technique. An innovative decomposition approach procedure is systematic and simple. The novel technique can apply to any type ordinary or partial differential equations.

REFERENCES